Advanced Topics in Mathematical Physics

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Dedicated to Cynthia and Mercedes

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Abstract.

Contents

Preface

This is the preface and it is created using a TeX field in a paragraph by itself containing `\chapter*{Preface}`. When the document is loaded, this appears if it were a normal chapter, but it is actually an unnumbered chapter.

Part 1

Foundations

CHAPTER 1

Set Theory

I assume that the reader has a good understanding of set theory material studied in his first course in freshman calculus. Here we develop a brief course on some advanced topics in set theory as it is taught to students of mathematics. We begin it with familiarity with union and intersection on class of sets. Then we discuss degenerate cases in sets which need to work with empty set.

1.1. Operations over a set

Definition 1.1.1. *Union over a Set : Union over a set is generalization of union of sets that fuses all members of a set together. These members are usually sets constitutes of other elements.*

$$\bigcup \mathcal{A} = \{x \mid \exists a \in \mathcal{A} \ni x \in a\}$$
$$\bigcup \varnothing = \varnothing$$

Definition 1.1.2. *Intersection Over a Set : Intersection over a set is generalization of intersection of sets that leaves out those elements common to all members of that set. These members are usually sets constitutes of other elements.*

$$\bigcap \mathcal{A} = \{x \mid \forall a \in \mathcal{A} \Rightarrow x \in a\}$$

In contrast to union, intersection over an empty set has no conclusive meaning and we have to define the intersection for the empty set as,

$$\bigcap \varnothing \triangleq \varnothing$$

1.2. Power Sets

I assume reader is already familiar with the idea of power set. To refresh that idea I define,

Definition 1.2.1. *Power Set : It is defined as the set of all subsets of any set $\mathcal{A}$. Precisely,*

$$\mathscr{P}\mathcal{A} = \{x \mid x \subseteq \mathcal{A}\}.$$

And also we have,

$$\mathscr{P}\varnothing = \{\varnothing\}$$

Hence, if $x \subseteq \mathcal{A}$ we, then, can write $x \in \mathscr{P}$. Please pay attention to belongness $\in$ symbol here. Also as an exercise note that $\bigcup \mathscr{P}\mathcal{A} = \mathcal{A}$.

Remark 1.2.1. *Important : We know what is a power set. Frequently we need to select certain collections of subsets of a set with certain structure out of the entire collection of subsets. For example $\mathfrak{M}$ which is a subcollection of $\mathscr{P}$. That is, $\mathfrak{M} \subseteq \mathscr{P}$. When we freely select an arbitrary collection and like to impose certain structure to them we call that collection a free collection and we show it by $\mathscr{F}$. To impose the certain structure to this collection $\mathscr{F}$, we make an intersection over* *all those collections that have that structure and contain $\mathscr{F}$ as a subset. Then we have the* ***smallest*** *collection shown say by $\mathscr{F}^*$ that is endowed with our desired structure. We easily can verify that having any two sets in $\mathscr{F}^*$ then we have their intersection in $\mathscr{F}^*$. Also if a set belongs to $\mathfrak{F}^*$ then all of its subsets also belong to $\mathscr{F}^*$.*

1.3. Empty Set Degenerate Cases

It is interesting to chase the last remark of the previous section in building further sets by getting the power sets of empty set, such as

$$\mathscr{P}\mathscr{P}\varnothing \quad \text{or} \quad \mathscr{P}\mathscr{P}\mathscr{P}\varnothing \quad \text{or} \quad \mathscr{P}\mathscr{P}\mathscr{P}\mathscr{P}\varnothing \ ...$$

A clever way of making more sets is that to assum that we have a beginning set S_0 and make a pyramid of sets using only S_0 and power sets made thereby such that each lower step includes the higher step, in this way

$$\begin{gathered} S_0 \\ S_1 = S_0 \cup \mathscr{P}S_0 \\ S_2 = S_1 \cup \mathscr{P}S_1 \\ \vdots \\ S_{n+1} = S_n \cup \mathscr{P}S_n \\ \vdots \end{gathered}$$

You can notice that,

$$S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_{n+1} \subset \cdots$$

Please note that the number of elements in each set S_{n+1} is finitely limited. We could continue building up any set with diabolically any number of elements but still we have a finite set. To overcome that we make the following set,

$$S_\omega = S_0 \cup S_1 \cup S_2 \cup \cdots \cup S_{n+1} \cup \cdots$$

That set has countably infinite number of elements

1.3.1. Natural Number System.

The exciting application of above observation is building natural number system from the beginning empty set $\varnothing$. We define,

$$\begin{aligned} 0 &= \{\} \\ &= \varnothing \end{aligned}$$

0-set has no element. Then

$$\begin{aligned} 1 &= 0 \cup \mathscr{P}0 \\ &= \varnothing \cup \{\varnothing\} \\ &= \{\{\varnothing\}\} \end{aligned}$$

1-set has one element. Then

$$\begin{aligned} 2 &= 1 \cup \mathscr{P}1 \\ &= \{\{\varnothing\}\} \cup \{\varnothing, \{\varnothing\}\} \\ &= \{\varnothing, \{\varnothing\}\} \end{aligned}$$

2-set has two elements. And

$$\begin{aligned} 3 &= 2 \cup \mathscr{P}2 \\ &= \{\varnothing,\ \{\varnothing\}\} \cup \{\varnothing,\ \{\varnothing,\ \{\varnothing\}\},\ \varnothing,\ \{\varnothing\}\} \\ &= \{\varnothing,\ \{\varnothing\},\ \{\varnothing,\ \{\varnothing\}\}\} \end{aligned}$$

3-set has three elements. And finally,

$$\vdots$$

$$n + 1 = n \cup \mathscr{P}n$$

n-set has n elements. Here again we can make the infinite set of natural numbers shown by ω.

$$\omega = 0 \cup 1 \cup 2 \cup 3 \cup \cdots \cup n \cup \cdots$$

Definition 1.3.1. *Arimathic Addition : Addition of two natural numbers m and n is shown by symbol + and defined as,*

$$m + n \triangleq m \cup n.$$

Arimathic multiplication is just an addition repeated many times.

$$m.n \triangleq \underbrace{m \cup m \cup \cdots \cup m}_{for\, n\, times.}$$

1.4. Mappings

An ordered pair is a set theory concept defined carefully to be constructed out of two elements of sets such that if the position of elements in their arrangements changes their assigned meaning changes. An ordered pair of two elements $x \in X$ and $y \in Y$ is shown with notation (x, y). More concisely,

Definition 1.4.1. *Ordered Pair : An ordered pair is a set such that,*

1. *Generally $(x, y) \neq (y, x)$ for $x \in X$ and $y \in Y$.*
2. *If $(x, y) = (v, w)$ for some x , $v \in X$ and some y , $w \in Y$, then $x = v$ and $y = w$.*

It is interesting to know that in abstract mathematics an ordered pair (x, y) is defined as equal to the set $\{\{x\}, \{x, y\}\}$ such that it satisfies uniquely the requirements of the above definition.

Remark 1.4.1. *: Please look at these interesting corollaries. You can convince yourself.*

$$\bigcup (x,\ y) = \{x,\ y\}$$
$$\bigcup (X \times Y) = X \cup Y$$
$$\bigcap (x,\ y) = \{x\}$$
$$\bigcap (X \times Y) = X$$

Definition 1.4.2. *Cartesian Products : Cartesian product of two sets X and Y is the set of all* ***ordered*** *pairs $\{\forall\ (x, y)\,|x \in X \text{ and } y \in Y\}$. In particular we can define the Cartesian product of $X \times X$.*

Remark 1.4.2. *: Cartesian product of more than two sets such as X, Y, and Z can be defined in a similar fashion with an important warning: this product is not generally associative, except that we have Cartesian product of the same set as follows.*

We can extend the Cartesian product to $X \times X \times \cdots \times X \times X$ such that we can define X^n for a set X.
There are degenerate cases worth of some attention.

(1) The product $X \times \varnothing$. This is equal to the set of singletons $(x,\ x)$. To see that we write, $(x,\ x) = \{\{x\}, \{x, x\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$. Now, one element of product $X \times \varnothing$ can be written as $\{\{x\}, \{x,\ \}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$. We left a blank space after x, to show that we have selected an element from empty set as the second component of the ordered pair. We can call this as an **ordered single** and show it by (x). Note that this is $\{\{x\}\}$ and is different from $\{x\}$.
(2) The product $\varnothing \times \varnothing$. This is considered special case of the above and is equal to the set $\{\{\ \}\} = \{\varnothing\}$. We know that this is not an empty set anymore and is different from just an empty set $\{\ \} = \varnothing$. We can call it an **ordered empty** and show it by $(\ )$ or in some contexts by Λ.
(3) The product $\varnothing \times X$. This, if you prefer, can only be defined as the empty set $\varnothing$. We cannot find anything out of the set $\{\varnothing,\ \{x\}\}$. It has not any any meaning.
(4) In X^n is it possible to have $n = 0$?
Answer is affirmative. We define it as the set of all ordered empties, $X^0 \triangleq \{(\ )\}$[**1**]. Hence, $X^0 = \{\varnothing \times \varnothing\}$. We show $X^0 = \{\Lambda\}$, and we call it empty product. This Λ set notation later will get other usages.
(5) In X^n is it possible to have $n = 1$?
Answer is affirmative. We define $X^1 = \{(x)\}$, the set of all ordered singles. Hence, $X^1 = X \times \varnothing$
(6) In X^n is it possible to have $n = \infty$ countably?
Answer is again affirmative. We study such products later.
(7) In X^n is it possible to have $n = \infty$, but uncountable?
Answer is again affirmative. We later define the Cartesian product for any value of n.

Definition 1.4.3. *:*

Definition 1.4.4. *Relation : A relation R from a set X to another set Y is just any subset of $X \times Y$. When $x \in X$ is related through R to $y \in Y$ we show it as xRy, or $(x,\ y) \in R$.*

Note that the subset mentioned in above definition is selected on an arbitrary appropriateness. That is, any arbitrary selection of any subset of $X \times Y$ is said to be a relation from X to Y.
Mapping is a relation R from a set X to another set Y, where for each element of X we assign only one unique element in Y. Instead of letter R, we prefer to usually use the letter f for a mapping. More concisely,

Definition 1.4.5. *Mapping : A mapping f of set X to set Y is a relation from X to Y such that $\forall x \in X$ there is only a unique $y \in Y$ that satisfies xfy or $(x,\ y) \in f$. We use notations $f : X \longrightarrow Y$, reads as mapping f from X to Y, and $x \mapsto y$ reads*

x maps to y for showing a mapping. Finally, we write $y = f(x)$ and read it as y is the map of x under f. Usually we write them in a stacked form as,

$$f : X \longrightarrow Y$$
$$x \mapsto y$$
$$y = f(x)$$

*The last notation is also read as y is the **value** of f at x. In, yet another wording we say f takes x to y. We say y is **the** image of x under mapping f and x is **a** pre-image of y with respect to f.*

A mapping frequently is termed as a **function**. We keep the usage of term "function" for mappings to the set of **real** numbers $\mathbb{R}$, or the set of **complex** numbers $\mathbb{C}$, or its subsets all through this book. Hence we call a **real valued** or a **complex valued** mapping a real function or a complex function, respectively. Later we see that a mapping from a vector space to its *scalar* field is usually called a **functional**. This term is kept as it is, due to historical usage, though it could be avoided.

Remark 1.4.3. *: To clear the meaning of **unique** in definition of mapping we can, **alternatively** say that if we take two different elements y_1 such that $y_1 = f(x_1)$ and y_2 such that $y_2 = f(x_2)$ and we find out that $y_1 \neq y_2$ then to have a mapping it is necessary that $x_1 \neq x_2$. If any point x in X maps to more than one point y in Y then f is **not** a mapping anymore. It will be a relation. This is how we check if a relation is a mapping. Note that it is possible in a mapping that different x's in X map to the same y in Y, inverse is not true. To summarize, assume*
$y_1 = f(x_1)$ and $y_2 = f(x_2)$ then,

$$if\, y_1 \neq y_2 \qquad then \quad x_1 \neq x_2,$$

alternatively,

$$if\, x_1 = x_2 \qquad then \quad y_1 = y_2;$$

on the other hand,

$$if\; y_1 = y_2 \quad \text{then it could be that} \quad x_1 \neq x_2,$$

alternatively,

$$if\; x_1 \neq x_2 \quad \text{then it could be that} \quad y_1 = y_2;$$

Definition 1.4.6. *Domain of a Mapping : Assume $f : X \longrightarrow Y$ is a mapping, then the set X is defined as the **domain** of the mapping f.*

Definition 1.4.7. *Co-domain of a Mapping : Assume $f : X \longrightarrow Y$ is a mapping, then the set Y is defined as the **co-domain** of the mapping f.*

In a mapping f the set of those points $y \in Y$ such that y is the image of some points $x \in X$ constitute a subset $A \subseteq Y$. This subset is the range of the mapping. More concisely, we have,

Definition 1.4.8. *Range of a Mapping : Assume $f : X \longrightarrow Y$ is a mapping, then the set $\forall y \in Y$ such that $\exists x \in X$ and $y = f(x)$ is a subset $A \subseteq Y$. This subset is the **range** of the mapping and is shown by notation $f[X]$.*

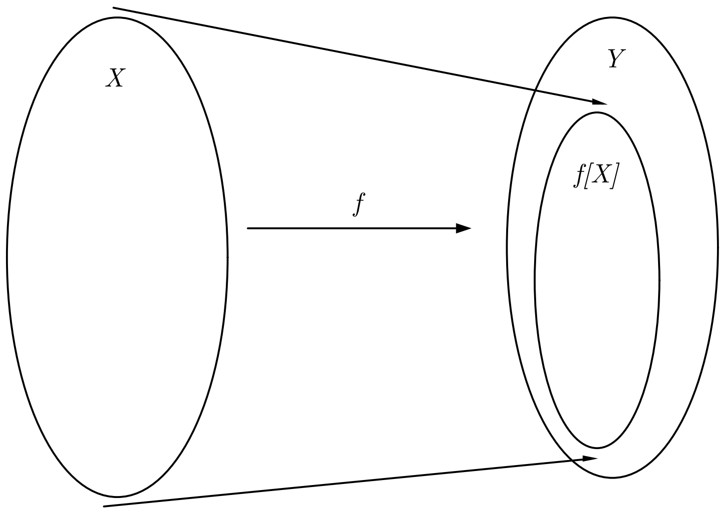

FIGURE 1. Range of Set X under Mapping f into Set Y.

Range of a mapping f is also called the **set of values** of the mapping f.

Definition 1.4.9. *Image of a Mapping : Assume $f : X \longrightarrow Y$ is a mapping, and $E \subseteq X$ then the set $\forall y \in Y$ such that $x \in E$ and $y = f(x)$ is a subset $S \subseteq Y$. This subset is the **image** of the set E and is shown by notation $f[E]$.*

Definition 1.4.10. *Inverse Image of a Mapping : Assume $f : X \longrightarrow Y$ is a mapping, and $S \subseteq Y$ then the set $\forall x \in X$ such that $y \in S$ and $y = f(x)$ is a subset $E \subseteq X$. This subset E is the **inverse image** of the set S and is shown by notation $f^{-1}[S]$. We frequently, call $f^{-1}[S]$ as the **pre-image** of S.*

If $y \in Y$ then you should recognize between $x = f^{-1}(y)$ where $x \in X$ and $E = f^{-1}[\{y\}]$ where $E \subseteq X$. Though it is true that frequently $f^{-1}(y)$ gives more than one $x \in X$ and some people might have that in mind when they interchangeably use the same notation for both $f^{-1}(y)$ and $f^{-1}[\{y\}]$.
You might notice that f^{-1} is not necessarily a mapping for itself, and it could be only a relation not a mapping.

Definition 1.4.11. *Composition of Mappings : Assume X and Y and Z are three sets and $W \subseteq Y$. Further, consider mappings $f : X \longrightarrow Y$ and $g : W \longrightarrow Z$. If the range $f[X]$ of f has common elements with domain W of g, that is, if $f[X] \cap W \neq \varnothing$ then we can define **composite** mapping of f and g shown as $g \circ f$ or simply gf by $gf : X \longrightarrow Z$.*

Hence f takes x to the y and then g takes y to the z, such that overall gf takes x to the z.

Definition 1.4.12. *Surjective Mappings : Assume $f : X \longrightarrow Y$ is a mapping. If for each $y \in Y$ we can find **at least** one (i.e., one or more than one) elements $x \in X$ such that $y = f(x)$ then we say the mapping f is **surjective**. A surjective mapping is called a **surjection**.*

In other words, in a surjective mapping the range of f coincides with the codomain Y.

Definition 1.4.13. *Onto Mapping : This is another term for a* ***surjective*** *mapping.*

We say a surjection maps X **onto** the set Y, and if possible we write $f : X \xrightarrow{onto} Y$ or $f : X \xrightarrow{sur} Y$ or $f : X \twoheadrightarrow Y$. When a mapping is not known to be surjective we say f maps X **into** the Y, like this $f : X \xrightarrow{into} Y$.

Remark 1.4.4. *: We understand that the range of* f *is the image set* $f[X]$. *Hence in a surjective mapping, we have* $Y = f[X]$.

Remark 1.4.5. *: Further it is always possible to restrict the codomain of a mapping such that the mapping* f *converts to a surjective mapping. This restriction is different with restriction of mapping in its domain. The surjective mapping then will be* $f : X \longrightarrow f[X] \subseteq Y$. *Still, we use the same notation* f *for our surjective mapping produced in this way.*

This restriction later will be of some use in understanding topological embedding.
To show that a mapping is surjective we should take each y in the co-domain Y and check if there exists an x in domain X such that we can ensure that the selected y is the image of that x. We express this test in mathematical form as,

$$\forall y \in Y \exists x \in X \ni y = f(x)$$

Definition 1.4.14. *Saturated Sets : Assume* $f : X \longrightarrow Y$ *is a surjection and* $A \subset X$. A *is called a* ***saturated*** *subset of* X with respect *to* f *if* A *contains every subset* $f^{-1}[y]$ *that intersects with* A.

In another word, A is equal to inverse image $f^{-1}[B]$ of some subset $B \subset Y$.

Remark 1.4.6. *: If* A *is not saturated with respect to* f *then generally,* $A \subset f^{-1}[f[A]]$. *But for a saturated subset* A *we have* $A = f^{-1}[f[A]]$.

Remark 1.4.7. *: Later we see that if* f *is a continuous map, a saturated set helps to define a* ***quotient*** *map and a quotient topology.*

Definition 1.4.15. *Injective Mappings : Assume* $f : X \longrightarrow Y$ *is a mapping. If for each* $y \in Y$ *we can find* ***only*** *one (i.e.,* ***not*** *more than one) elements* $x \in X$ *such that* $y = f(x)$ *then we say the mapping* f *is* ***injective***. *An injective mapping is called an* ***injection***.

Remark 1.4.8. *: In an injective mapping* $f : X \longrightarrow Y$ *if* $x_1 \neq x_2$ *when* $x_1, x_2 \in X$ *then we have* $f(x_1) \neq f(x_2)$. *This is a way that you can check a mapping is an injection.*

Definition 1.4.16. *One-one Mapping : Sometimes is termed as one-to-one mapping, this is another term for an* ***injective*** *mapping. We show an injection as* $f : X \xrightarrow{inj} Y$ *or* $f : X \xrightarrow{1-1} Y$ *or* $f : X \rightarrowtail Y$.

An idea similar to saturated sets can be explored to defining co-saturated sets.

Definition 1.4.17. *Co-saturated Sets : Assume* $f : X \longrightarrow Y$ *is an injection and* $B \subset Y$. B *is called a* ***co-saturated*** *subset of* Y with respect *to* f *if* B *contains every subset* $f[x]$ *that intersects with* B *where* $x \in A$ *and* $A \subset X$.

In another word, B is equal to image $f[A]$ of some subset $A \subset X$.

Remark 1.4.9. *: If A is not co-saturated with respect to f then generally,* $ff^{-1}[B] \subset B$. *But for a saturated subset A we have* $ff^{-1}[B] = B$.

Remark 1.4.10. *: Later we see that if f is a continuous map, a co-saturated set helps to define an **induced** map and an induced topology.*

Saturated and co-saturated sets help us to keep in mind properties of mappings defined on the intersection of sets with respect to the intersection of their images and also inverse image of intersections.

Remark 1.4.11. *: Assume* $f : X \longrightarrow Y$ *and* $A \subset X$ *and* $B \subset X$ *and* $E \subset Y$ *and* $F \subset Y$ *then, generally we have,*

(1)
$$if \quad A \subseteq B \quad then \quad f(A) \subseteq f(B)$$

(2)
$$f(A \cup B) = f(A) \cup f(B)$$

(3)
$$f(A \cap B) \subseteq f(A) \cap f(B)$$

(4)
$$f(A - B) \supseteq f(A) - f(B)$$

(5)
$$f(C(A)) \supseteq C(f(A))$$

(6)
$$f(A - B) \subseteq f(A)$$

(7)
$$if \quad E \subseteq F \quad then \quad f^{-1}(E) \subseteq f^{-1}(F)$$

(8)
$$f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$$

(9)
$$f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$$

(10)
$$f^{-1}(E - F) = f^{-1}(E) - f^{-1}(F)$$

(11)
$$f^{-1}(C(E)) = C(f^{-1}(E))$$

Now we define a bijective mapping where all subsets of domain and co-domain are saturated and co-saturated respectively for the mapping.

Definition 1.4.18. *Bijective Mappings : A bijective mapping is one that is both a surjective mapping and an injective mapping. A bijective mapping is also is said to be a **bijection**. We show an injection as* $f : X \xrightarrow{bi} Y$ *or as* $f : X \xrightarrow[onto]{1-1} Y$ *or* $f : X \leftrightarrow Y$.

Frequently a bijection might be referred as a **one to one correspondence**. One important bijection on a finite set is **permutation**.

Definition 1.4.19. *Permutation : A bijection* σ *on a finite set S is said to be a permutation on S. Hence,* $\sigma : S \xrightarrow{bi} S$.

Definition 1.4.20. *Operator : Operator is a mapping where domain is a Cartesian product of a set* $X \times X$ *and co-domain is the set* X *itself. An operator mapping is called an operation.*

Definition 1.4.21. $n-$*ary Operator : Is defined as a mapping from cartesian product* X^n ***into*** *the set* X*. That is,* $f : X^n \longrightarrow X$.

Remark 1.4.12. *Degenerate Case : It is worth noting the degenerate case of* $n = 0$*, where, you can remember,* X^n *defined to be* $X^0 \triangleq \{()\}$*. In this case we assume* $f(\{\varnothing\})$ *is a member of* X*. This operator,* $f : X^0 \longrightarrow X$ *is called a* ***nulary*** *operator. An examples of a nulary operator is the 0 element and the 1 element in a Boolean algebra (Section 2.4).*

Remark 1.4.13. *: Similarly, we have* ***unary*** *operator* $f : X^1 \longrightarrow X$*. An example of a unary operator is negation of an integer in additive group of integers, as later we study.*

Definition 1.4.22. *Pre-set of a Set : Assume* X *and* Y *are sets. Then the set of all mappings from* X *into* Y*, that is,* $\{\forall f | f : X \longrightarrow Y\}$ *is said to be the pre-set of set* X*. We show the pre-set by* ${}^XY$ *notation.*

Definition 1.4.23. $f-$*dual Set of a Set : Assume* X *is a set. Then the set of all mappings from* X *into* X*, that is,* $\{\forall f | f : X \longrightarrow X\}$ *is said to be the* $f-$*dual set of set* X*. We show this set by* X^{*f}.

This X^{*f} is the same as pre-set ${}^XX$.
We should study some degenerate cases for mappings before we should move further.

(1) Mapping $f : \varnothing \longrightarrow Y$. You can reason that this mapping is just the empty set $f = \varnothing$
(2) Mapping $f : \varnothing \longrightarrow \varnothing$. We can decide that this one is a special case of above and hence, $f = \varnothing$.
(3) Mapping $f : X \longrightarrow \varnothing$. This has no meaning. We do not define a mapping with **non-**empty domain and an empty co-domain. It means as if you have a mapping and then you do not assign anything to members of its domain. But mapping means assigning something to domain. Hence a contradiction.
(4) Pre-set ${}^{\varnothing}Y$. This has only one member. So, ${}^{\varnothing}Y = \{\varnothing\}$.
(5) Pre-set ${}^{\varnothing}\varnothing$. This is special case of above and ${}^{\varnothing}\varnothing = \{\varnothing\}$.
(6) Pre-set ${}^X\varnothing$. We might decide that this set is just an empty set, i.e., ${}^X\varnothing = \varnothing$.

Definition 1.4.24. *Power Set Mapping : Assume* $f : X \longrightarrow Y$ *is a mapping. We can define a mapping shown as* 2^f *from power set* 2^Y *to power set* 2^X*, that is,* $2^f : 2^Y \longrightarrow 2^X$ *such that if* $S \in 2^Y$ *then* $2^f(S) = f^{-1}(S)$.

Definition 1.4.25. *Restriction of a Mapping : Assume mapping* $f : X \longrightarrow Y$ *is defined. Then if we have a subset* $A \subseteq X$ *we can define the mapping* $g : A \longrightarrow Y$ *such that* $g(x) = f(x), \forall x \in A$ *as the restriction of* f *to the subset* A*. We show restriction of* f *to* A *by* $f|_A \stackrel{\Delta}{=} g$ *notation.*

Definition 1.4.26. *Extension of a Mapping : Assume mapping* $f : X \longrightarrow Y$ *is defined. Then if we have a set* $\Omega \supseteq X$ *we can define the mapping* $g : \Omega \longrightarrow Y$ *such that* $g(x) = f(x), \forall x \in X$ *as the extension of* f *to the set* Ω*. We show extension of* f *to* Ω *by* $f|^{\Omega}$ *notation.*

Definition 1.4.27. *Submodulus Set : Assume mapping $f : X \longrightarrow X$ is defined. Then if $f[X] \subset X$, we say X is* ***submodulus set*** *of f.*

Definition 1.4.28. *Modulus Set : Assume mapping $f : X \longrightarrow X$ is defined. Then if $f[X] = X$, that is if f is a surjection, we say X is* ***modulus set*** *of f.*

Definition 1.4.29. *Identity Mapping : assume mapping $id : X \longrightarrow X$ is defined such that, for $x \in X$ we have $x \longmapsto x$, or $id(x) = x$ then id is called the identity mapping. We show the identity mapping on a set X by id_X.*

Definition 1.4.30. *Inclusion Mapping : Assume we have a subset $A \subseteq X$ then the restriction of the identity mapping $id : X \longrightarrow X$ to the subset A, that is $id_X|_A$ is said to be the inclusion mapping of A and is shown as ι_A.*

Definition 1.4.31. *Embedding Mapping : Assume we have a set $\Omega \supseteq X$ then the extension of the identity mapping $id : X \longrightarrow X$ to the set Ω, that is $id_X|^{\Omega}$ is said to be the embedding mapping of X into Ω. X is said as embedded in Ω.*

Frequently, embedding might be spelled as imbedding. It is better not to contrast this two different spelling as different concepts. Note that this 1.4.31 is one **basic** interpretation of embedding. This definition opens the way for later and further understanding of more sophisticated usage of the notion.

Remark 1.4.14. *: We notice that we can have two types of restriction and two types of extension for a mapping $f : X \longrightarrow Y$.*

(1) *Restriction of domain X.*
(2) *Extension of domain X.*
(3) *Restriction of co-domain Y.*
(4) *Extension of co-domain Y.*

Often it is helpful to show composition of mappings in diagrams. It is generalization of arrow notation we already have used. For example, assume we have composition gf of g and f, with restriction we already imposed on the domain of g. We can show it by the following diagram.

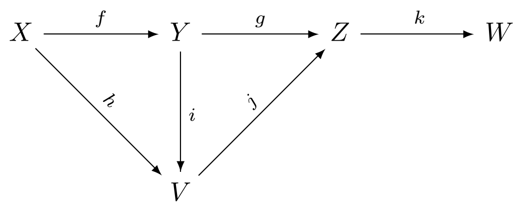

FIGURE 2. A Commutative Diagram.

You notice that from any domain to another co-domain there is one or more than one path consisting of one or many mappings. For example from X to V we have h or alternatively, through Y we have the composition if. From X to Z we have three paths gf and jh and jif.

Definition 1.4.32. *Commutative Diagrams : A commutative diagram shows equivalent composition of mappings on equivalent* ***paths****.*

Definition 1.4.33. *Axiom of Choice : For any relation R from a set X to another set Y there is a mapping $f : X \longrightarrow Y$.*

Definition 1.4.34. *Inverse Mapping : Assume $f : X \overset{inj}{\longrightarrow} Y$. Then the mapping $g : f[X] \overset{inj}{\longrightarrow} X$ is called **the** inverse mapping of mapping f.*

We show the inverse mapping of f with f^{-1}. You may notice that f is also inverse of the mapping $f^{-1} : f[X] \overset{inj}{\longrightarrow} X$.

Remark 1.4.15. *: It can is easily oberved that $f \circ f^{-1} = id_X$ and also $f^{-1} \circ f = id_{f[X]}$.*

We can have the following diagram.

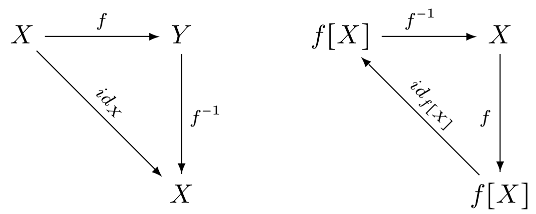

FIGURE 3. Commutative Diagrams for Inverse Mappings.

Definition 1.4.35. *Invertible Mapping : A bijective mapping is said to be an invertible mapping.*

Definition 1.4.36. *Iteration of Mapping : Assume $f : X \overset{inj}{\longrightarrow} X$. Then X is a submodulus set with respect to f. We can define iterations of f by induction as,*

$$f^0(x) = x, \quad and \quad f^{n+1}(x) = f \circ f^n(x), \quad \forall x \in X \quad and \quad \forall n > 0.$$

If f is a bijection then X is a modulus set with respect to f and we can extend this definition to negative integers, as well, by defining,

$$f^{n-1}(x) = f^{-1} \circ f^n(x), \quad \forall x \in X \quad and \quad \forall n \leq 0.$$

Further we can define,

$$f^{n+m}(x) = f^m \circ f^n(x), \quad \forall x \in X \quad and \quad \forall n, m \in \mathbb{Z}.$$

Definition 1.4.37. *Function : A function is a mapping from any set X to the set of real numbers $\mathbb{R}$ or real $n-$tuples $\mathbb{R}^n$. That is $f : X \longrightarrow \mathbb{R}^n \quad \forall n \in \mathbb{N}$.*

Definition 1.4.38. *Kernel of a Function : Kernel is the set of all element in X whose images in $\mathbb{R}$ is the single element zero $0 \in \mathbb{R}$. In other words, kernel is the pre-image $f^{-1}[\{0\}]$ of singleton $\{0\} \subseteq \mathbb{R}^n$.*

later we define kernel for mappings into the certain other algebraic dtructures besides $\mathbb{R}$ when there is a neutral element 0 defined on them.

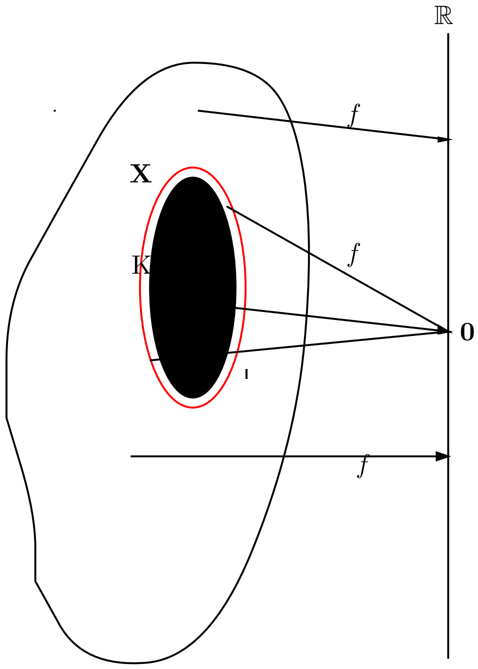

FIGURE 4. Kernel of function f is everything on red boundary and inside it.

CHAPTER 2

Structures on sets

2.1. Relations

Definition 2.1.1. *Reflexive Relation : for each x we have $x\mathcal{R}x$.*

An example of such relation is less than or equal relation ($\leq$) on set of , say integers. For any integer x we have $x \leq x$.

Definition 2.1.2. *Irreflexive Relation : There is **not** any x such that $x\mathcal{R}x$ holds.*

For example the relation less than ($<$) on set of integers does **not** hold. For any integer x it is **not** true that $x < x$.

Definition 2.1.3. *symmetric Relation : $x\mathcal{R}y$ implies $y\mathcal{R}x$*

An example is the equality relation $x = y$

Definition 2.1.4. *Anti-symmetric Relation: $x\mathcal{R}y$ does **not** imply $y\mathcal{R}x$ but if $x\mathcal{R}y$ and $y\mathcal{R}x$ **both** hold then $x = y$.*

A familiar example of anti-symmetric relation is the less than ($<$) relation $x < y$.

Definition 2.1.5. *Transitive Relation: $x\mathcal{R}y$ and $y\mathcal{R}z$ implies $x\mathcal{R}z$.*

Again the less than ($<$) relation $x < y$ satisfies transitivity requirement as a relation.

Definition 2.1.6. *Trichotomy Relation : **Only one** of the three relations $x\mathcal{R}y$, $y\mathcal{R}x$ or $x = y$ holds.*

Definition 2.1.7. *Equivalence Relation : This relation is reflexive, symmetric and transitive.*

2.2. Order in Sets

The order relation on sets is shown by the usual notation $\leq$ instead of $\mathcal{R}$.

Definition 2.2.1. *Dominant set : Assume we have sets X and Y. We say Y is dominant over X, if there exists an injection f from X into the Y. That is, $f : X \xrightarrow{inj} Y$. We show this relation by $X \preccurlyeq Y$.*

If Y is dominant over X then we have $f[X] \subseteq Y$ and $card(X) \leq card(Y)$.

Definition 2.2.2. *Preordered sets : A set is preordered when there is a reflexive and transitive relation on some of the elements in the set.*

Definition 2.2.3. *Partially ordered sets : A set is partially ordered when there is a reflexive and **anti**symmetric, and transitive relation on **some** of the elements in the set.*

Definition 2.2.4. *Totally ordered sets : A set is totally ordered when there is a reflexive and* ***anti****symmetric, and transitive relation on* ***all*** *of the elements in the set.*

Definition 2.2.5. *Chain : Any totally ordered subset of a partially ordered set.*

Definition 2.2.6. *Notation : Assume we have ordered set X ordered with relation R, we show it by notation $\mathfrak{O}\,(X,\ R)$.*

Definition 2.2.7. *Similarity : Assume we have ordered sets $\mathfrak{O}\,(X,R)$ and $\mathfrak{O}\,(Y,S)$. The mapping $f : X \longrightarrow Y$ is said to be a* ***similarity*** *if from the relation x_1Rx_2 $\forall x_1, x_2$ in domain X, we have $f\,(x_1)\,Sf\,(x_1)$.*

Definition 2.2.8. *Order Preserving Mapping : Let $f : X \longrightarrow Y$ and $\forall x, \xi \in X, x \leq \xi$ then $f(x) \leq f(\xi)$.*

You notice that order preserving mapping is a similarity. This will be of use when we study notion of **embedding** in context of universal algebra.

2.3. Lattice and Well Ordering

Assume A is an ordered subset of an ordered set Ω. An element x in Ω is a lower bound for elements of A whenever for all elements a in A we have $x \leq a$. In mathematical notation we write

Definition 2.3.1. *Lower Bound : Let $A \subseteq \Omega$, then an element $x \in \Omega$ is a lower bound for A whenever $\forall\ a \in A$ we have $x \leq a$.*

Assume A is an ordered subset of an ordered set Ω. An element x in Ω is an upper bound for elements of A whenever for all elements a in A we have $a \leq x$. In mathematical notation we write

Definition 2.3.2. *Upper Bound : Let $A \subseteq \Omega$, then an element $x \in \Omega$ is an upper bound for A whenever $\forall\ a \in A$ we have $a \leq x$.*

Assume x and y are two elements belong to an ordered set Ω. We say y covers x if $x \leq y$, and additionally you cannot find a z that comes between x and y in their order. In mathematical notation we write

Definition 2.3.3. *Cover : Let Ω be a partially ordered set and $x,\ y \in \Omega$. We say y covers x whenever $x \leq y$ and there does not exist z such that $x \leq z \leq y$.*

Definition 2.3.4. *Least Upper Bound : Let $A \subseteq \Omega$, then an element $x \in \Omega$ is the least upper bound for A whenever x is an upper bound for$A$ and if y is another upper bound for A then $x \leq y$.*

Definition 2.3.5. *Greatest Lower Bound : Let $A \subseteq \Omega$, then an element $x \in \Omega$ is the greatest lower bound for A whenever x is a lower bound for$A$ and if y is another lower bound for A then $y \leq x$.*

Definition 2.3.6. *Maximum (Greatest) element : Let $A \subseteq \Omega$; further, if $x \in \Omega$ is the least upper bound of A then $x \in A$.*

Definition 2.3.7. *Minimum (Least) element : Let $A \subseteq \Omega$; further, if $x \in \Omega$ is the greatest lower bound of A then $x \in A$.*

Definition 2.3.8. *Join : Least upper bound of a set with two members a and b is called join of a and b we show it by $a \vee b$.*

Definition 2.3.9. *Meet : Greatest lower bound of a set with two members a and b is called join of a and b we show it by* $a \wedge b$ *.*

Definition 2.3.10. *Directed partially ordered sets: Every pair of elements have an upper bound.*

Definition 2.3.11. *Well-ordered sets : Every non-empty subset has a least element.*

Definition 2.3.12. *Lattice : A lattice* $\mathcal{L}$ *is a partially ordered set such that,*

(1) *Every pair of elements of* $\mathcal{L}$ *have a joint and a meet*
(2) *There are element* $0 \in \mathcal{L}$ *and element* $1 \in \mathcal{L}$ *such that for every element* $a \in \mathcal{L}$ *we have* $0 \leq a \leq 1$

Definition 2.3.13. *Complement : Assume* $\mathcal{L}$ *is a lattice, and* $a \in \mathcal{L}$*. Now, consider you can find* $a' \in \mathcal{L}$ *such that it satisfies the following conditions:*

(1) $a' \vee a = 1$.
(2) $a' \wedge a = 0$.

Then we say a' *is* ***the complement*** *of* a*. We use symbol* $a^{'}$ *to show complement of element* $a \in \mathcal{L}$*.*

It is trivial to see that if a' is the complement of a, then a is also complement of a'; that is, $(a^{'})^{'} = a$

Definition 2.3.14. *Orthocomplemention Mapping: This is defined as mapping* ω *on lattice* $\mathcal{L}$ *as* $\omega : \mathcal{L} \longrightarrow \mathcal{L}$ *such that for any* $a \in \mathcal{L}$ *we have* $a \mapsto a^{'}$*, that is, it maps element* $a \in \mathcal{L}$ *to its complement* $a^{'}$ *Further,*

(1) $(a^{'})^{'} = a$.
(2) *if* $a \leq b$ *then we have* $b^{'} \leq a^{'}$.

Definition 2.3.15. *Orthocomplemented Lattice : A lattice with an orthocomplemented mapping defined in it is called orthocomplemented*

Definition 2.3.16. *Complete Lattice : When every non-empty subset of a lattice have least upper bound and greatest lower bound, the lattice is complete.*

Definition 2.3.17. *Distributive Lattice : regarding join and meet element for comparing three elements of a lattice L*

(1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

2.4. Boolean Algebra

Definition 2.4.1. *Boolean Algebra : An orthocomplemented, distributive lattice is called a Boolean algebra.*

Definition 2.4.2. *Infinite distributive :*

(1) $a \bigwedge S = \bigwedge \{a \vee b \mid b \in S\}$
(2) $a \bigvee S = \bigvee \{a \wedge b \mid b \in S\}$

Definition 2.4.3. *Locale/Frame : is a lattice with infinite distributive property.*

2.5. Partition

Assume Ω is a set and $\mathfrak{M}$ is a collection of subsets of Ω.

Definition 2.5.1. *Filter or Up-set : For any* $A \in \mathfrak{M}$ *and* $A \supseteq B$ *then* $B \in \mathfrak{M}$

Definition 2.5.2. *Ideal or Down-set : For any* $A \in \mathfrak{M}$ *and* $A \subseteq B$ *then* $B \in \mathfrak{M}$

Definition 2.5.3. *Complete Family : For any* $A \in \mathfrak{M}$ *and* $B \in \mathfrak{M}$ *then* $A \cap B \in \mathfrak{M}$

Definition 2.5.4. *Anti-chain : For any* $A_i \in \mathfrak{M}$ *and* $A_j \in \mathfrak{M}$ *then* $A_j \subsetneq A_i$

Definition 2.5.5. *Chain : For any* $A_i \in \mathfrak{M}$ *and* $A_j \in \mathfrak{M}$ *then either* $A_i \subseteq A_j$ *or* $A_j \subseteq A_i$ *is correct.*

Definition 2.5.6. *Maximal Chain :* $\mathscr{C}$ *is a maximal chain if for any other chain* $\mathscr{C}'$ *if* $\mathscr{C} \subseteq \mathscr{C}'$ *then* $\mathscr{C} = \mathscr{C}'$

Definition 2.5.7. *Partition (Finite): The class of sets* $\{ A_j \}$ *such that* $A_i \subseteq \Omega\,;\, i = 1, 2, \cdots, n$ *and* $A_i \cap A_j = \varnothing$ *when* $i \neq j$ *and* $\Omega = \bigcup_{i=1}^{n} A_i$ *.*

Definition 2.5.8. *Partition (Countable): The class of sets* $\{ A_j \}$ *such that* $A_i \subseteq \Omega\,;\, i = 1, 2, \cdots$ *and* $A_i \cap A_j = \varnothing$ *when* $i \neq j$ *and* $\Omega = \bigcup_{i=1}^{\infty} A_i$ *.*

Definition 2.5.9. $\mathfrak{M}$*–partition : The class of subsets* $\{A_j\}$ *such that*

(1) $A_i \in \mathfrak{M}$.
(2) $\{A_j\}$ *is a partition.*

Definition 2.5.10. *Dissection : Dissection is the less common word for partition.*

Definition 2.5.11. *Base of Sets :*

Definition 2.5.12. *Net of Sets :*

Remark 2.5.1. *Important (Repeat Remark 1.2.1): We know what is a power set. Frequently we need to select certain collections of subsets of a set with certain structure out of the entire collection of subsets. For example* $\mathfrak{M}$ *which is a subcollection of* $\mathscr{P}$*. That is,* $\mathfrak{M} \subseteq \mathscr{P}$*. When we freely select an arbitrary collection and like to impose certain structure to them we call that collection a free collection and we show it by* $\mathscr{F}$*. To impose the certain structure to this collection* $\mathscr{F}$*, we make an intersection over all those collections that have that structure and contain* $\mathscr{F}$ *as a subset. Then we have the* ***smallest*** *collection shown say by* $\mathscr{F}^*$ *that is endowed with our desired structure. We easily can verify that having any two sets in* $\mathscr{F}^*$ *then we have their intersection in* $\mathfrak{F}^*$*. Also if a set belongs to* $\mathscr{F}^*$ *then all of its subsets also belong to* $\mathscr{F}^*$*.*

Remark 2.5.2. *Axiom of Choice: Every partially ordered set has a maximal chain.*

2.6. Quotient Set

Definition 2.6.1. *Equivalence Set (Class) : Assume* R *is an equivalence relation on set* X*. Choose an element* $\xi \in X$*. We show the set of all elements* $x \in X$ *that are related to* ξ*, that is,* $\xi R x$ *by symbol* $[\,\xi\,]_R$ *and call it equivalence set (class) of element* ξ*, conveniently written as* $[\,\xi\,]$*.*

Definition 2.6.2. *Identification : When we have an equivalence set* $[\,\xi\,]$ *for a* $\xi \in X$*, we say this set is an* ***identification*** *of all elements* $x \in X$ *related to* ξ *through* R*. All elements* $x \in X$ *are identified by set* $[\,\xi\,]_R$*.*

Note that when you identify a set of elements $x \in X$ by $[\,\xi\,]_R$, then you have exhausted all those elements to the single class $[\,\xi\,]_R$. Each $[\,\xi\,]_R$ is one element of a superset, say W, which is different from the set X. X is not a subset of W and W is not a subset of X.

Exercise 1. *An easy exercise : If $\xi \neq \zeta$, then either $[\,\xi\,]_R \cap [\,\zeta\,]_R = \varnothing$ or $[\,\xi\,]_R = [\,\xi\,]_R$.*

Definition 2.6.3. *Identifying Function (Kuratowski) : Assume f is a mapping of X to Y, that is $f : X \longrightarrow Y$. Take $\xi \in X$ and define relation R on X as $xR\xi$ for $x \in X$ if and only if $f(x) = f(\xi)$. We say that the mapping f* ***identifies*** *a class of elements of X .*

It easy to prove that this relation is an equivalence relation on X. Given $\xi \in X$, the set of all $x \in X$ equivalent modulo f to ξ, that is, $[\xi]_f$, sometimes is called as the **orbit** of ξ under f.

Remark 2.6.1. *: Kuratowsky identification actually is defined when we define relation R on X as $xR\xi$ for $x \in X$ if and only if $f^m(x) = f^n(\xi)$ for $\xi \in X$ and some $n, m \in \mathbb{Z}$, where X is a modulus set with respect to mapping f. Orbit of ξ is shown as $[\xi]_f^{m,n}$, and reads as orbit of ξ with respect to f* ***of order*** *m, n.*

Example 2.6.1. *: Let define $b : \mathbb{Z} \longrightarrow \{0, 1\}$ such that $b(z) = 0$ whenever z is even, and otherwise, $b(z) = 1$ when z is odd. Hence even integers are identified by 0 through the mapping b and odd integers by 1. Then orbit of, say 7, i.e., $[7]_b$ is the set of all odd numbers.*

Definition 2.6.4. *Take ξ in X and assume X is a submodulus set with respect to the mapping $f : X \longrightarrow X$. Then $[\xi]_f^{0,1}$, that is the set of all $x = f^k(\xi)$ are called fixed points of f of order k.*

Definition 2.6.5. *Quotient Set X/R : Assume R is an equivalence relation in X. For each $\xi \in X$, there is an equivalence set $[\,\xi\,]_R$. Then the set $\{[\,\xi\,]_R | \forall \xi \in X\}$ is called the* ***quotient set*** *of X* ***modulo*** *R, shown as X/R.*

- Each equivalence set $[\,\xi\,]_R$ exhausts certain subset of X to a single class. The set of all these classes is the quotient set X/R.
- Quotient set X/R makes a partition on set X.
- In the Example 2.6.1 above we show the quotient set as $\mathbb{Z}/f$.

Definition 2.6.6. *Canonical map π : The map $\pi : X \longrightarrow X/R$, where $\xi \mapsto [\,\xi\,]_R$ is called canonical map on X.*

The canonical map π is surjective.

Remark 2.6.2. *: In reference to definition 2.6.3, we can define a mapping $\kappa : \mathbb{Z}/f \longrightarrow f(Y)$ that partitions $f(Y)$ into equivalence classes. This should be contrasted clearly with the canonical mapping, 2.6.6, defined above.*

Example 2.6.2. *Let $0 \le \xi \le 1$. Identify all $x \in \mathbb{R}$ by relation ($\xi R x$, when, $x = k + \xi$), for all integers $k \in \mathbb{Z}$. Quotient set X/R is the unit circle $\mathbb{S}^1$, or the unit circle in the complex plane. Each $[\,\xi\,]_R$ is one point on the circle and each addition of the integer k rotates once round the circle.*

In discussing degenerate cases of empty sets, we succeded to built all whole numbers from zser to any arbitrary large number. We know natural numbers are natural in the sense that human started counting with them. This exclude zero from the set of natural numbers. Later human discovered zero and after that negative number. Having natural numbers in hand, how can we build zero amnd negative integers from them without additional material? The only thing we need is definition of Cartesian product, and from it the definition of an equivalence relation. We show the set of positive (natural) numbers by $\mathbb{N}$

Example 2.6.3. *Set of Integers $\mathbb{Z}$: Take $(\mu,\nu) \in \mathbb{N} \times \mathbb{N}$ and define relation shown with symbol $\sim$ by identifying all $(m,n) \in \mathbb{N} \times \mathbb{N}$ as $(m,n) \sim (\mu,\nu)$ if and only if $m+\nu = n+\mu$. Then the equivalence set (class) $[(\mu,\nu)]$ is said to be an integer. The set of all integers is shown by $\mathbb{Z}$. Note that $0 \triangleq [(\mu,\mu)]$.*

Definition 2.6.7. *Congruence Relation : Take two integers a and b. If their difference is an integer multiple of some non-zero integer k, that is, $a-b = m.k$, we say a and b are related **modulo** n and we show it by $a \equiv b\,(modulo\,n)$. We read this as a is **congruent** to b modulo n.*

In another word, we know two congruent integers as equal, or rather equivalent numbers. Hence all congruent numbers (**modulo** n) exhaust or are identified by a single class $[\quad]_{\equiv(modulo\ n)}$.

Definition 2.6.8. $\mathbb{Z}_n$ *: Take $\zeta \in \mathbb{Z}$, the set of all $z \in \mathbb{Z}$ such that $z \equiv \zeta\,(modulo\,n)$ (that is, $z - \zeta = k.n$) is the equivalent class $[\zeta]_{\equiv(modulo\ n)}$.*
We define $\mathbb{Z}_n = \left\{[\zeta]_{\equiv(modulo\ n)} \mid \zeta \in \mathbb{Z}\right\}$

After now we show $[\zeta]_{\equiv(modulo\ n)}$ simply as $[\zeta]_n$.

Definition 2.6.9. *:*

Example 2.6.4. $\mathbb{Z}_1$ *: For each $\zeta \in \mathbb{Z}$ we should take all $z \in \mathbb{Z}$ such that $\zeta \equiv z\,(modulo\ 1)$ (that is, $\zeta - z = k.1$).*
Assume $\zeta = 2$ and we like to construct $[\,2\,]_{\equiv(modulo 1)}$, or $[\,2\,]_1$. Then the set of all $z \in \mathbb{Z}$ such that $z - 2 = k$, or $z = 2 + k$, $\forall k \in \mathbb{Z}$ coincides with the set of integers $\mathbb{Z}$. Hence, $[\,2\,]_1 = \{\cdots,-3,-2,-1,0,+1,+2,+3,\cdots\} = \mathbb{Z}$. The same results for any $\zeta \in \mathbb{Z}$ other than 2. Hence, $[\,0\,]_1 = [\,1\,]_1 = [\,2\,]_1 = \cdots$. Then we have, $\mathbb{Z}_1 = \{[\,0\,]_1\}$.

Please note to distinguish between $\mathbb{Z}$ and $\mathbb{Z}_1$. Set $\mathbb{Z}$ has countably infinite members, while $\mathbb{Z}_1$ is a singleton set, has only **one** element $[\,0\,]_1$. Incidentally, we observe that $[\,0\,]_1 = \mathbb{Z}$ and also $\mathbb{Z}_1 = \{\mathbb{Z}\}$, and $\mathbb{Z}_1 \neq \mathbb{Z}$.

Example 2.6.5. $\mathbb{Z}_2$ *: For each $\zeta \in \mathbb{Z}$ we should take all $z \in \mathbb{Z}$ such that $\zeta \equiv z\,(modulo\ 2)$ (that is, $\zeta - z = k.2$).*
Assume $\zeta = 5$ and we like to construct $[\,5\,]_{\equiv(modulo 2)}$, or $[\,5\,]_2$. Then the set of all $z \in \mathbb{Z}$ such that $z - 5 = k.2$, or $z = 5 + k.2$, $\forall k \in \mathbb{Z}$ coincides with the set of all odd integers. Hence, $[\,5\,]_2 = \{\cdots,-7,-5,-3,-1,+1,+3,+5,+7\cdots\}$. The same results for any other odd integer $\zeta \in \mathbb{Z}$ other than 5 such as 1 or 9 or 23. On the other hand, if we select an even integer, say, $\zeta = 6$ and construct $[\,6\,]_{\equiv(modulo 2)}$, or $[\,6\,]_2$. Then the set of all $z \in \mathbb{Z}$ such that $z - 6 = k.2$, or $z = 6 + k.2$, $\forall k \in \mathbb{Z}$ will be the set of all even integers. Hence, $[\,6\,]_2 = \{\cdots,-8,-6,-4,-2,0,+2,+4,+6,+8\cdots\}$. The same results for any other even integer $\zeta \in \mathbb{Z}$ other than 6 such as 2 or 10 or 18.

Therefore, $[\,0\,]_2 = [\,2\,]_2 = [\,4\,]_2 = \cdots$ *and* $[\,1\,]_2 = [\,3\,]_2 = [\,5\,]_2 = \cdots$. *Then we have,* $\mathbb{Z}_2 = \{[\,0\,]_2\,,[\,1\,]_2\}$.

$\mathbb{Z}_2$ has **two** members. All even integers are identified with one class $[\,0\,]_2$ and all odd integers are identified with another class $[\,1\,]_2$.

Example 2.6.6. $\mathbb{Z}_3$ *: For each* $\zeta \in \mathbb{Z}$ *we should take all* $z \in \mathbb{Z}$ *such that* $\zeta \equiv z\,(modulo\ 3)$ *(that is,* $\zeta - z = k.3$*).*
Assume $\zeta = 4$ *and we construct* $[\,4\,]_{\equiv(modulo 3)}$*, or* $[\,4\,]_3$*. Then the set of all* $z \in \mathbb{Z}$ *such that* $z - 4 = k.3$*, or* $z = 4 + k.3$*,* $\forall k \in \mathbb{Z}$ *coincides with the set of all integers in form of,* $[\,3\,]_3 = \{\cdots, -12, -9, -6, -3, 0, +3, +6, +9, +12\cdots\} = \{0 + 3.k | \forall k \in \mathbb{Z}\}$*. If we select any integer* ζ *equal to one of these integers the equivalence set would be equal to that again. We select* $[\,0\,]_3$ *as representative of this class.Now select an integer* $\zeta = 1+3.k$*, say,* $\zeta = 4$ *and construct* $[\,4\,]_3$*. Then the set of all* $z \in \mathbb{Z}$ *such that* $z-4 = k.3$*, or* $z = 4+k.3$*,* $\forall k \in \mathbb{Z}$ *will be the set* $[\,4\,]_3 = \{\cdots, -8, -5, -2, +1, +4, +7, \cdots\} = \{1 + 3.k | \forall k \in \mathbb{Z}\}$*. Select* $[\,1\,]_3$ *as identification of these integers. A similar argument for* $\zeta = 2+3.k$ *such as* $\zeta = 5$ *results in* $[\,5\,]_3 = \{\cdots, -4, -1, +2, +5, +8, \cdots\}$*. Choose* $[\,2\,]_3$ *for this set. At the end we have,* $\mathbb{Z}_3 = \{[\,0\,]_3\,,[\,1\,]_3\,,[\,2\,]_3\}$*.*

$\mathbb{Z}_3$ has **three** members.

- A general observation is that $card\,(\mathbb{Z}_n) = n$. That is $\mathbb{Z}_n$ has n members.
- Another observation is that $\bigcup \mathbb{Z}_n = \mathbb{Z}$
- Always $\mathbb{Z}_n \notin \mathbb{Z}$ and $\mathbb{Z}_n \not\supseteq \mathbb{Z}$.
- It is easily could be seen that there is a bijection between set $\mathbb{Z}_n$ and set $\{0,\ 1,\ 2,\ \ \dots,\ n-1\}$.

Remark 2.6.3. $n\mathbb{Z}$ *: Here it is a good point to become familiar with the set* $n\mathbb{Z}$ *and contrast it carefully with* $\mathbb{Z}_n$*. It is a countably infinite subset of* $\mathbb{Z}$ *and is defined as, assuming* $n \neq 0$*,*

$$n\mathbb{Z} = \{\ \dots,\ -3n,\ -2n,\ -n,\ 0,\ +n,\ 2n,\ 3n,\ \dots\}$$

That is, each integer is multiplied by an n, forming a subset of $\mathbb{Z}$*. For example,*

$$2\mathbb{Z} = \{\ \dots,\ -6,\ -4,\ -2,\ 0,\ +2,\ 4,\ 6,\ \dots\}$$

and,

$$5\mathbb{Z} = \{\ \dots,\ -15,\ -10,\ -5,\ 0,\ +5,\ 10,\ 15,\ \dots\}$$

and,

$$12\mathbb{Z} = \{\ \dots,\ -36,\ -24,\ -12,\ 0,\ +12,\ 24,\ 36,\ \dots\}$$

We always have,

$$n\mathbb{Z} \subseteq \mathbb{Z}$$

Please note that elements of $n\mathbb{Z}$ *are not multiplications. In essence, they are results of* ***additions*** *of each member of* $\mathbb{Z}$ *added n times together.*

Remark 2.6.4. *Sometimes, you may encounter with a notation conveniently written as*

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$$

This means that you identify points of $\mathbb{Z}$ *modulo n. In group theory this notation is helpful in recognizing the quotient groups.*

In Example 2.6.3 we succeeded to build set of integers $\mathbb{Z}$ by defining an equivalence relation on $\mathbb{N} \times \mathbb{N}$. Here, we follow that approach and build the set of rational numbers $\mathbb{Q}$ from $\mathbb{Z} \times \mathbb{Z}$. Again we remove zero from $\mathbb{Z}$ and show the resulting set by $\mathbb{Z} = \mathbb{Z} \setminus \{o\}$.

Example 2.6.7. *Set of Rational Numbers $\mathbb{Q}$: Take $(\mu, \nu) \in \mathbb{Z} \times \mathbb{Z}$ and define relation shown with symbol $\sim$ by identifying all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ as $(m, n) \sim (\mu, \nu)$ if and only if $m.\nu = n.\mu$. Then the equivalence set (class) $[(\mu, \nu)]$ is said to be a rational number. The set of all integers is shown by $\mathbb{Z}$. Note that $[(\mu, \mu)] = 1$.*

2.7. Indexing Sets

Definition 2.7.1. *Indexing Function : Let $\mathfrak{M}$ be a collection of subsets of the set Ω. A surjective map f from a set J, called **index set**, to $\mathfrak{M}$ is said to be an indexing function.*

Definition 2.7.2. *Net :*

Definition 2.7.3. *Indexed Family of Sets : Let $\mathfrak{M}$ be a collection of subsets of the set Ω. Take the set J as an index set. Then the indexing surjection $f : J \longrightarrow \mathfrak{M}$ of $\alpha \mapsto f(\alpha)$ together with the family $\mathfrak{M}$ is called an indexed family of sets. We denote $f(\alpha)$ by A_α. Indexed family is shown by$\{A_\alpha\}_{\alpha \in J}$*

A J-tuple is very similar to an indexed family of sets. The difference is that it is defined on members of a set rather than subsets of a set.

Definition 2.7.4. *$J-$tuple : Assume X is a subset of the set Ω. Take the set J as an index set. Then a mapping $\mathbf{x} : J \longrightarrow X$ of $\alpha \mapsto \mathbf{x}(\alpha)$ is called a $J-$tuple. We denote $\mathbf{x}(\alpha)$ by x_α and we call it the $\alpha-$coordinate of $\mathbf{x}$. The collection shown by $\{x_\alpha\}_{\alpha \in J}$ is called coordinates of $\mathbf{x}$.*

Definition 2.7.5. *$J-$power : The set of all $J-$tuples of set X is called a $J-$power of X and is denoted by ${}^J X$ or X^J. $J-$power is a subset of the power set $\mathscr{P}(J \times X)$*

Definition 2.7.6. *Cartesian Product : Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of subsets of Ω. Assume $X = \bigcup_{\alpha \in J} A_\alpha$. Then the cartesian product $\prod_{\alpha \in J} A_\alpha$ is the set $\{\mathbf{x} : J \to \bigcup_{\alpha \in J} A_\alpha\}$ such that $\mathbf{x}(\alpha) \in A_\alpha$, $\forall \alpha \in J$*

Definition 2.7.7. *Box : Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of subsets of Ω. Further assume that there exists the Cartesian product $\prod_{\alpha \in J} A_\alpha$. Now assume $\forall \alpha \in J$ there exists $B_\alpha \subseteq A_\alpha$ Then the box B is defined as $\prod_{\alpha \in J} B_\alpha$*

A parameterizing function become important when we define a manifold.

Definition 2.7.8. *parameterizing Function : Assume A is a parameter set. Consider the indexed family of sets $\{A_\alpha\}_{\alpha \in J}$and an indexed family of functions $\{f_\alpha\}_{\alpha \in J}$ such that $f_\alpha : A \longrightarrow A_\alpha$ then the function $f : A \longrightarrow \prod_{\alpha \in J} A_\alpha$ where $f(\alpha) = (f_\alpha(\alpha))_{\alpha \in J}$ is called a parametrising function on A.*

Definition 2.7.9. *Projection Map : Take $\{A_\alpha\}_{\alpha \in J}$ as an indexed family of sets. The mapping $\pi_\beta : \prod_{\alpha \in J} \longrightarrow A_\beta$ where $\pi_\beta\left((a_\alpha)_{\alpha \in J}\right) = a_\beta$ is called a projection map on A.*

Idea of bundles come from idea of product space.

Definition 2.7.10. *Product Space : Again let* $\{A_\alpha\}_{\alpha\in J}$ *be an indexed family of subsets of* Ω. *We can define* $S_\beta = \pi_\beta^{-1}\left((B_\beta)_{\beta\in J}\right)$ *for some* $B_\beta \subseteq A_\beta$ *and for all* $\beta \in J$. *Then* $S = \bigcup_{\beta\in J} S_\beta$ *is called the product space of the indexed family* $\{A_\alpha\}_{\alpha\in J}$.

Definition 2.7.11. *Disjoint Union : Take* $\{A_\alpha\}_{\alpha\in J}$ *as a collection of indexed family of sets. Then the set* $\coprod A_\alpha = \bigcup_{\alpha\in J}\{(x,\alpha)\,|\,\forall a \in A_\alpha\}$ *is called disjoint product of the collection of sets.*

2.8. Cut

So Far we built natural, integer, and rational numbers. Using a **cut** we are going to build *real* numbers.

Definition 2.8.1. *Dedekind Cut : A dedekind cut is defined as a set* $\mathbf{x}$ *such that,*

(1) *It is a nonempty subset of* $\mathbb{Q}$, *that is,* $\varnothing \subsetneq \mathbf{x} \subsetneq \mathbb{Q}$.
(2) *The set* $\mathbf{x}$ *is closed downward, that is,* $\forall s \in \mathbf{x}$ *if* $r < s$ *then* $r \in \mathbf{x}$.
(3) $\forall s \in \mathbf{x}$, $\exists t \in \mathbf{x}$ *such that* $s < t$.

CHAPTER 3

Measure Theory Structures

Assume Ω is a set and $\mathfrak{M}$ is a collection of subsets of Ω.

3.1. Semi-Ring Structures on Sets

Definition 3.1.1. *Von Neumann Semi-ring : A collection of subsets $\mathfrak{M}$ is a Von Neuman semi-ring if it satisfies two following conditions*

(1) *If $A, B \in \mathfrak{M}$ then $A - B \in \mathfrak{M}$.*
(2) *If $A, B \in \mathfrak{M}$ and $A \subseteq B$ then there is a chain $\{A_i | A_i \in \mathfrak{M}, i = 0, 1, \cdots, n\}$ such that $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = B$ and $A_i - A_{i-1} \in \mathfrak{M}$ for $i = 1, \cdots, n$.*

Definition 3.1.2. *Semi-ring : A collection of subsets $\mathfrak{M}$ is a semi-ring if and only if*

(1) *If $A, B \in \mathfrak{M}$ then $A - B \in \mathfrak{M}$.*
(2) *If $A, B \in \mathfrak{M}$ then for partition $\{ A_i \,| A_i \in \mathfrak{M}, i = 0, 1, \cdots\}$ such that $A - B = \bigcup_{i=1}^{\infty} A_i$.*

3.2. Ring Structures on Sets

Definition 3.2.1. *Ring (Boolean Ring or Finite Union Ring): A ring is defined as a collection of subsets where,*

(1) *If $A, B \in \mathfrak{M}$ then $A - B \in \mathfrak{M}$.*
(2) *If for any collection $\{ A_i \,| A_i \in \mathfrak{M}, i = 0, 1, \cdots, n\}$ we have $\bigcup_{i=1}^{n} A_i \in \mathfrak{M}$.*

Definition 3.2.2. *$\sigma-$Ring (Infinite Union Ring): A $\sigma-$ring is defined as a collection of subsets where,*

(1) *If $A, B \in \mathfrak{M}$ then $A - B \in \mathfrak{M}$.*
(2) *If for any countable collection $\{ A_i \,| A_i \in \mathfrak{M}, i = 0, 1, \cdots\}$ we have $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$.*

Definition 3.2.3. *$\sigma - \mathfrak{M}(Ring)$: It is a collection $\mathfrak{M}$ of subsets of Ω where*

(1) *$\mathfrak{M}$ is a $\sigma-$ring.*
(2) *$\exists \{ A_i \,| A_i \in \mathfrak{M}, i = 0, 1, \cdots\}$ such that $\Omega = \bigcup_{i=1}^{\infty} A_i$*

3.3. Field Structures on Sets

Definition 3.3.1. *Field (Kuratovsky Field): A collection $\mathfrak{M}$ of subsets of Ω is a field whenever,*

(1) *$\mathfrak{M}$ is a ring and*
(2) *$\Omega \in \mathfrak{M}$*

Definition 3.3.2. *$\sigma-$Field (Borel Field): A collection $\mathfrak{M}$ of subsets of Ω is a $\sigma-$field whenever,*

(1) $\mathfrak{M}$ *is a* $\sigma-$*ring and*
(2) $\Omega \in \mathfrak{M}$

Definition 3.3.3. $\sigma - \mathfrak{M}(Field)$ *: It is a collection* $\mathfrak{M}$ *of subsets of* Ω *where*

(1) $\mathfrak{M}$ *is a* $\sigma-$*field.*
(2) $\exists \{ A_i \,|A_i \in \mathfrak{M}, i = 0, 1, \cdots\}$ *such that* $\Omega = \bigcup_{i=1}^{\infty} A_i$

3.4. Algebra Structures on Sets

Definition 3.4.1. *Algebra (Boolean Algebra): An algebra is defined as a collection of subsets where,*

(1) *If* $A \in \mathfrak{M}$ *then* $A \in \mathfrak{M}$.
(2) *If for any finite collection of subsets* $\{ A_i \,|A_i \subseteq \Omega, i = 0, 1, \cdots, n\}$ *of* Ω *where* $\bigcup_{i=1}^{n} A_i \in \mathfrak{M}$ *then we have each* $A_i \in \mathfrak{M}$.
(3) $\Omega \in \mathfrak{M}$

Definition 3.4.2. $\sigma-$*Algebra : A* $\sigma-$*algebra is defined as a collection of subsets where,*

(1) *If* $A \in \mathfrak{M}$ *then* $A^c \in \mathfrak{M}$.
(2) *If for any countable collection of subsets* $\{ A_i \,|A_i \subseteq \Omega, i = 0, 1, \cdots\}$ *of* Ω *where* $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$ *then we have each* $A_i \in \mathfrak{M}$.
(3) $\Omega \in \mathfrak{M}$

Definition 3.4.3. $\sigma - \mathfrak{M}(Algebra)$ *: It is a collection* $\mathfrak{M}$ *of subsets of* Ω *where*

(1) $\mathfrak{M}$ *is a* $\sigma-$*algebra.*
(2) $\exists \{ A_i \,|A_i \in \mathfrak{M}, i = 0, 1, \cdots\}$ *such that* $\Omega = \bigcup_{i=1}^{\infty} A_i$

3.5. Sequences of Sets

Definition 3.5.1. *Monotone Sequence:*

Definition 3.5.2. *Monotone Class : A monotone class is defined as a countable collection* $\mathfrak{M}$ *of subsets* $\{ A_i \,|A_i \subseteq \Omega, i = 0, 1, \cdots\}$ *where,*

(1) *If* $A_i \subseteq A_{i+1}$ *(or alternatively* $A_i \supseteq A_{i+1}$*) for each* $i = 0, 1, \cdots$*, and*
(2) $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$ *(or alternatively* $\bigcap_{i=1}^{\infty} A_i \in \mathfrak{M}$*).*

Definition 3.5.3. *Limit of a Sequence :* $\mathbb{R}$

CHAPTER 4

Algebraic Structures

4.1. Rudiments

Definition 4.1.1. *Binary Operation on a Set : Assume S is a set. A binary operation on two elements $a \in S$ and $b \in S$ is a mapping on $S \times S$. We show a binary operation by $\mathfrak{O}$ or $\circ$ or $\bullet$ or $\cdot$ or $*$. The result of operation is shown as $\mathfrak{O}(a, b)$ or $a \circ b$ or $a \bullet b$ or $a \cdot b$ or $a * b$, respectively. We choose $a \circ b$ as our standard notation for our binary operations. When there is no ambiguity we even prefer to use ab as the result of binary operation $\mathfrak{O}(a, b)$.*

Definition 4.1.2. *Structure : When we have a binary operation $\mathfrak{O}$ on set S, we say that the binary operation defines, or actually recognizes and reveals a structure $[S, \mathfrak{O}]$ on set S.*

Definition 4.1.3. *Closed : If the result of a binary operation on a set S is a member of the set S then we say the set S is **closed** under that specific binary operation.*

Definition 4.1.4. *Commutative : When $a \circ b$ and $b \circ a$ result in the same, we say that the binary operation is **commutative** and write it as $a \circ b = b \circ a$.*

Definition 4.1.5. *Associative : When $(a \circ b) \circ c$ and $a \circ (b \circ c)$ result in the same, we say that the binary operation is **associative** and write it as $(a \circ b) \circ c = a \circ (b \circ c) = a \circ b \circ c$.*

Definition 4.1.6. *Right Cancellation : We say the binary operation $\circ$ on set S follows the right cancellation rule if from the $a \circ c = b \circ c$ we can get to the $a = b$ for all the $a, b, c \in S$.*

Definition 4.1.7. *Left Cancellation : We say the binary operation $\circ$ on set S follows the left cancellation rule if from the $c \circ a = c \circ b$ we can get to the $a = b$ for all the $a, b, c \in S$.*

Definition 4.1.8. *Unity (neutral element): When in a set S for a binary operation $\circ$ we can find an element e such that $\forall s \in S$ we can have $e \circ s = s \circ e$ then we call e the **unity** or **neutral** element of S for binary operation $\circ$.*

Definition 4.1.9. *Unit (inverse element) : Assume $\circ$ is a binary operation defined in set S. Further assume that S is endowed with the unity element e. If for an $s \in S$ we can find an element u such that $u \circ s = s \circ u = e$. Then s is called a **unit** element of S. s and u are called **inverse** of each other.*

Next we assume that f is a mapping from set X to set Y. Also we have binary operation $\circ$ on X and binary operation $\bullet$ on Y

Definition 4.1.10. *Morphism (Structure Preserving Mapping) :A morphism* f *from set* X *to set* Y *carries structure of* X *into the structure of* Y*. That is,* $f(a \circ b) = f(a) \bullet f(b)$

(1) *HomoMorphism: This is a morphism from a set* X *into a different set* Y*.*
(2) *EndoMorphism: EndoMorphism is a morphism from a set* X *into the same set* X*.*

Definition 4.1.11. *Kernel of Morphism : Assume* f *is a morphism from set* X *to set* Y*. Then the set* $\forall x \in X$ *such that* $f(x) = 0$ *is called the kernel of morphism* f*.*

Definition 4.1.12. *Homomorphism : When the mapping* $f : X \longrightarrow Y$ *is just an* ***into*** *mapping.*

Definition 4.1.13. *Monomorphism : When that mapping is an injection mapping then the homomorphism is a monomorphism*

Definition 4.1.14. *Epimorphism : When the mapping is a surjection mapping then the homomorphism is an epimorphism.*

Definition 4.1.15. *Isomorphism : If the mapping is both an injection and a surjection then the homomorphism is an isomorphism.*

Definition 4.1.16. *Endomorphism : When the mapping* $f : X \longrightarrow X$ *is just an* ***into*** *mapping.*

Definition 4.1.17. *Endo-monomorphism: When that mapping is an injection mapping then the endomorphism is an endo-monomorphism*

Definition 4.1.18. *Endo-epimorphism : When the mapping is a surjection mapping then the endomorphism is an endo-epimorphism.*

Definition 4.1.19. *Automorphism (Endo-isomorphism): If the mapping is both an injection and a surjection then the endomorphism is an automorphism.*

4.2. Groups

Definition 4.2.1. *Groupoid : A set G with a binary operation defined on elements of that set is called a groupoid. A groupoid is a binary collection of the set G and the operator* $\mathfrak{O}$*; that is,* $(G, \mathfrak{O})$*.*

Definition 4.2.2. *Semi-group : Semi-group is a groupoid* $(G, \mathfrak{O})$ *where* $\mathfrak{O}$ *is* ***associative****.*

Definition 4.2.3. *Monoid : Monoid is a semi-group* $(G, \mathfrak{O})$ *where a* ***unity*** *element* $e \in G$ *exists for the binary operation* $\mathfrak{O}$*.*

Definition 4.2.4. *Group : Group is a monoid* $(G, \mathfrak{O})$ *where each element* $g \in G$ *is a* **unit** *with respect to the binary operation* $\mathfrak{O}$*.*

Definition 4.2.5. *Abelian Group (Commutative Group) : In group* $(G, \mathfrak{O})$ *the binary operation* $\mathfrak{O}$ *could be a commutative operator. In that case the group* $(G, \mathfrak{O})$ *is called a* ***commutative*** *or* ***Abelian*** *group*

Definition 4.2.6. *Subgroup : In group* $(G, \mathfrak{O})$ *let* $H \subset G$*. Assume* $a, b \in H$*. If* $a \circ b^{-1} \in H$*, as well, then* H *is a* ***subgroup*** *of* G*.*

Definition 4.2.7. *Subgroup Generated by a Subset : In group* $(G, \mathfrak{O})$ *let* $X \subset G$. X *is not necessarily a subgroup. Assume a set* Y *consists of all elements* p *formed from* n *elements* $x_1, x_2, \cdots, x_n$, *not necessarily different, taken from* X *such that* $p = x_1^{\epsilon_1} \circ x_2^{\epsilon_2} \circ \cdots \circ x_n^{\epsilon_n}$ *where* $\epsilon_j = \pm 1, \forall j = 1, \cdots, n$. *This new set* Y *has structure of a group and is called subgroup of* G *generated by* X. *We show this set by* $gp(X)$.

Definition 4.2.8. *Finitely Generated Subgroup : When* X *is a finite set with* n *elements the subgroup generated by* X *is called a* ***finitely*** *generated subgroup. This set, usually is shown by* C_n, *and is said to be* ***cyclic*** *group of order* n.

Definition 4.2.9. *Cyclic Subgroup : Assume the set* X *is a singleton* $\{x\}$, *then the finitely generated subgroup is called the* ***cyclic*** *group generated by* x. *We show it by* $gp(\{x\})$.

Definition 4.2.10. *Cosets : Let* H *be a subgroup of* G. *Take* $g \in G$ *The set of all elements formed as* $gh, \forall h \in H$ *is called the* ***left coset*** *of the subgroup* H *and is denoted as* gH.

Definition 4.2.11. *Cosets : Let* H *be a subgroup of* G. *Take* $g \in G$ *The set of all elements formed as* $hg, \forall h \in H$ *is called the* ***right coset*** *of the subgroup* H. *Right coset of* H *is shown as* Hg.

Definition 4.2.12. *Index of a Subgroup : Number of right cosets of* H *in* G *is said to be the index of* H *in* G *and is shown as* $[G : H]$.

Definition 4.2.13. *Quotient* ***of*** *Groups : Let* G *be a group and* H *a subgroup of* G. *A partition of* G *by left cosets of* H *is called* ***quotient*** *of group* G *by the subgroup* H. *We show the resultant partition by* G/H.

It can be proved that G/H is a partition for G.

Definition 4.2.14. *Normal Subgroup : Let* H *be a subgroup of* G. H *is called a normal subgroup when* $g^{-1}hg \in H, \forall g \in G$ *and* $\forall h \in H$. *To show* H *as a normal subgroup of* G *we use notation* $H \lhd G$.

Definition 4.2.15. *Simple Group : A group without a* ***proper normal*** *subgroup is a simple group.*

Definition 4.2.16. *Factor Group : Let* G *be a group and* H *a* ***normal*** *subgroup of* G. *One can prove that the resulting quotient* G/H *is a* ***group*** *and a subgroup of* G. G/H *is called the* ***factor*** *group.*

Definition 4.2.17. *Commutator : Let* $x, y \in G$, *where* G *is a group. Then* ***commutator*** w *of* x *and* y *is defined as* $w = x^{-1}y^{-1}xy$ *we show the* ***commutator*** *by the bracket notation* $w = [x, y]$.

Definition 4.2.18. *Commutator Subgroup : The subgroup* G' *of* G *generated by* $[x, y]$, *that is,* $gp([x, y])$ *is called the commutator subgroup of* G.

We show that G' is normal in G.

Definition 4.2.19. *Center of a Group : This is the set of those elements* $a \in G$ *such that* $\forall g \in G$ *we have* $ag = ga$. *We show the* ***center*** *of* G *by* $A(G)$.

Definition 4.2.20. *Centraliser : Let* $A \subset G$, *where* A *is not necessarily a subgroup of* G. *This time select those elements* $g \in G$ *such that* $\forall a \in A$ *we have* $ag = ga$. *This new set built by help of* A *is called the* ***centraliser*** *of* A *in* G *and is shown by* $C(A)$.

Definition 4.2.21. *Normaliser : Let $A \subset G$, where A is not necessarily a subgroup of G. Again select those elements $g \in G$ such that we have $Ag = gA$. This set built by help of A is called the **normaliser** of A in G and is shown by $N(A)$.*

Definition 4.2.22. *Normaliser of a subgroup H: Let $A \subset G$, where A is not necessarily a subgroup of G, and H be a subgroup of G. Select those elements $h \in H$ such that we have $h = h^{-1}Ah$. This set built by help of A and H is called the **normaliser** of A in subgroup H and is shown by $N_H(A)$.*

Definition 4.2.23. *Subnormal Series : In the chain $\{e\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ of group G we have $A_i \lhd A_{i+1}$*

Definition 4.2.24. *Factors of a Subnormal Series : Quotients A_{i+1}/A_i are said to be the factors of a subnormal series.*

Definition 4.2.25. *Upper Central Series : A chain of subgroups $\{e\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ of group G is an **upper central series** whenever A_1 is the center of G and A_{i+1}/A_i is the center of G/A_i*

In this case $A_{i+1} \lhd G$

Definition 4.2.26. *Nilpotent Group : Upper central series $\{e\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ of group G is a finite chain.*

Definition 4.2.27. *Solvable Group : In the subnormal series the index of H in G that is, $[A_{i+1} : A_i]$ is a prime depending on i .*

Definition 4.2.28. *Composition Series : In the subnormal series A_{i+1}/A_i is a **simple** group; that is, it is a group without any **proper** normal subgroup.*

Definition 4.2.29. *H-conjugates of Subsets of a Group : Assume T and S are any two subsets of G and H is a subgroup of G; such that there exists $h \in H$ that implies $h^{-1}Sh = T$. Then S and T are called H-conjugates subsets of group G.*

Definition 4.2.30. *Conjugate Subsets : Let T and S to be any two subsets of G. Further, assume that there exists $g \in G$ such that $g^{-1}Sg = T$. Then S and T are called conjugates subsets of group G.*

Definition 4.2.31. *p-group : When the order of a group G is a power of a prime number p; that is, $|G| = p^r$ where r is a positive integer.*

Definition 4.2.32. *Sylow$p$-subgroup :*

4.3. Action of a Group

Definition 4.3.1. *Action of a Group on a Set: Assume there is a mapping, shown with symbol $\odot$, such that $\odot : G \times X \longrightarrow X$. Mapping $\odot$ is called the action of G on set X.*

Definition 4.3.2. *G-set of a Set : Let $\odot$ be an action of group G on set X. We say X is G-set if $\forall x \in X$ we have*

1. *$e \odot x = x$ where e is unity in G.*
2. *$(g_1 * g_2) \odot x = g_1 \odot (g_2 \odot x)$; $\forall g_1$ and $\forall g_2 \in G$, where $*$ is the binary operation in G.*

*Elements of G are also called **operators** on X when the context requires it.*

Note that in both definitions above, we took two members from G and X respectively in that order and G is on the **left** of X. The result is an element in X. We equally could define an action of G on the **right** of X.

Definition 4.3.3. *X_g−set : Let $\odot$ be an action of group G on set X. A subset Γ of X is said to be X_g−set if $\forall x \in \Gamma$ we have $g \odot x = x$ where g is an element in G.*

Definition 4.3.4. *G_X−group : Let $\odot$ be an action of group G on set X. A subset H of G is said to be G_x−set or **isotropic** subgroup of G if $\forall g \in H$ we have $g \odot x = x$ where x is an element in X.*

4.4. Rings

Definition 4.4.1. *Array of Binary Operations : An array of binary operations is a set of two or more binary operations defined to create an algebraic structures. We show it with a bracket, e.g., like this $[*, \odot]$.*

Definition 4.4.2. *Ringoid : This is defined as a structure built out of a set R and a duet array $[+, \cdot\,]$ of binary operations such that*

(1) *$(R, +)$ is an Abelian (commutative) group*
(2) *$(R, \cdot\,)$ is a groupoid.*
(3) *$\cdot$ is distributive over $+$, both from the left and from the right. That is $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.*

We show the ringoid as $(R, [+, \cdot\,])$

We use **'addition'** and **'multiplication'**, respectively to name the binary operations $[+, \cdot\,]$.

Definition 4.4.3. *Ring : A ring is a ringoid $(R, [+, \cdot\,])$ where $(R, \cdot\,)$ is a semigroup.*

Definition 4.4.4. *Unitary Ring : A unitary ring is a ring $(R, [+, \cdot\,])$ where $(R, \cdot\,)$ is a monoid.*

In any of the above three structures we can define new structures such that the multiplication to become commutative. A more interesting case is when you impose a constraint for multiplication to have right (left) cancellation law as well.

Definition 4.4.5. *Integral Domain : An integral domain is a ring (ringoid, ring, or unitary ring) where $a \cdot b = 0$ implies either $a = 0$ or $b = 0$ or both $a = b = 0$.*

An integral domain usually is called a **domain**. Hence, as it is said, in an integral domain we have **no zero divisor**. If a and b are two nonezero elements of domain R then theeir productr ab is nonezero.

Definition 4.4.6. *Division Ring : A division ring is a unitary ring $(R, [+, \cdot\,])$ where $(R, \cdot)$ is a group.*

Definition 4.4.7. *Skew Field : This is an old fashioned name for the division ring.*

Note that with the definition of division ring it is implied that the division ring is a ring with **unit** elements for multiplication.

Definition 4.4.8. *Field : A field is a division ring $(R, [+, \cdot\,])$ where $(R, \cdot\,)$ is an abelian (commutative) group.*

You note that a field is two abelian groups $(R, +)$ and $(R, \cdot\,)$ interwoven together through the distributive law of the underlying ringoid.

4.5. Ideals

Before proceeding to ideals, it is instructive to become familiar with a subring. That makes us able to clearly contrast it with an ideal. Concept of ideals has roots in determinants, multiplication of polynomials and symmetric polynomials.

Definition 4.5.1. *Subring : A subring S of the ring R is a subset of R such that*

(1) $(S, +)$ *is a subgroup of Abelian group* $(R, +)$.
(2) *If* $a \in S$ *and* $b \in S$ *then* $a \cdot b \in S$.
(3) $1 \in S$.

- The first item above is equivalent to proposition that $\forall a,\ b \in S$ we have $a - b \in S$, which then implies $0 \in S$.

Ideals bring the concept of group cosets to multiplication in rings. During this discussion we do not assume that the ring R is a commutative ring. We also avoid using right or left. We believe context is clear. All notations are written "right" sided.

Definition 4.5.2. *Ideal : Let* $(R, [+, \cdot\,])$ *is a ring. An ideal I is a subset of R such that*

(1) $(I, +)$ *is a subgroup of Abelian group* $(R, +)$ *and*
(2) *If* $i \in I$ *and* $r \in R$ *then* $i \cdot r \in I$.
(3) *Usually* $1 \notin I$.

- The first item (1) above is equivalent to proposition that $\forall a,\ b \in I$ we have $a - b \in I$, which then implies $0 \in I$.
- The second item (2) above is equivalent to proposition that $I.r \subseteq I, \forall r \in R$.
- R is an ideal in R.
- R is the **only** ideal in R that is a subring. No other ideal is a subring of R.
- $\{0\}$ is an ideal in R.
- The intersection of any family of ideals in R is an ideal in R.
- The third item (3) above is mentioned to help the reader to compare an ideal with a subring. There is only one ideal that defies this restriction as we see in the next definition.
- You might have noticed that a ring as we have define is not commutative in its multiplicative $(R, \cdot\,)$ semi-group. Hence we can have different left and right ideals. That conciseness is most of the time beyond the rigour considered for our exposure of the subject. If necessary we will mention it explicitly. We have tried to be consistent all the time.

Definition 4.5.3. *Proper Ideal : An ideal I in a ring R is a proper ideal if $I \neq R$.*

It is routine to show that in a proper ideal I, we always have $1 \notin I$
An ideal which is $\{0\}$ or R is said to be a **trivial** ideal. Otherwise, it is a **non-**trivial ideal.

Definition 4.5.4. *Simple Ring: A ring R which has* ***no*** *ideal but trivial ideals $\{0\}$ and R is called a simple ring.*

Hence a simple **ring** has only trivial **ideals**.

4.6. Arithmetic of Ideals

Ideals are a crucial point in understanding of a large part of algebra. We continue to make ourselves more friendly with concepts surrounding them. Though a bit artificial, arithmetic of ideals is important pedagogically and perhaps not much of later usage. As sets ideals have unions and intersection and other set theory operation that readers can work them out.
Addition is the set of term-by-term additions

Definition 4.6.1. *Addition of Ideals : Assume* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are ideals. Then we define addition of* $\mathfrak{a}$ *and* $\mathfrak{b}$ *as,*

$$\mathfrak{a} + \mathfrak{b} = \{a + b \mid a \in \mathfrak{a} \; and \, b \in \mathfrak{b}\}$$

Definition 4.6.2. *Multiplication of Ideals : Assume* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are ideals. Then we define multiplication of* $\mathfrak{a}$ *and* $\mathfrak{b}$ *as,*

$$\mathfrak{a}\mathfrak{b} = \left\{ \sum_{i=1}^{n} a_i b_i \mid a_i \in \mathfrak{a} \; and \, b_i \in \mathfrak{b}, \; n \in \mathbb{Z}^+ \right\}$$
$$= \{a_1 b_1, \; a_1 b_1 + a_2 b_2, \; a_1 b_1 + a_2 b_2 + a_3 b_3, \; \ldots\}$$

Definition 4.6.3. *Ideal Generated by a Subset of a Ring : Assume* X *is any subset of ring* R*. Consider the family of all ideals* $I_\alpha, \alpha \in J$ *in* R *such that* $X \subset I_\alpha, \forall \alpha \in J$*. Then* $\bigcap_{\alpha \in J} I_\alpha$ *is an ideal in* R *and we have* $X \subset \bigcap_{\alpha \in J} I_\alpha$*. We call* $\bigcap_{\alpha \in J} I_\alpha$ *the ideal generated by* X *and show it as* (X)*.*

Definition 4.6.4. *Principal Ideal (Ideal generated by an element of a ring) : If* X *is a singleton set* $\{\rho\} \subset R$ *then* (ρ) *is called the* ***principal ideal*** *generated by* ρ*.*

One can show that $(\rho) = \{x | x = \rho r; \forall r \in R\} = \rho R$.

Remark 4.6.1. *: You immediately appreciate that* $\rho = 1$ *cannot generate a proper ideal.*

Remark 4.6.2. *Scaling of Ring :* (ρ) *is the* ***scale*** *up of the ring* R*. That is* $(\rho) = \rho R$*. You can bring the idea of* ***translate*** *from cosets of a subgroup to here to create* ***quotient rings****.*

Similarly if $\{a_1, \cdots, a_n\}$ is a subset of R then

$$(a_1, \cdots, a_n) = \{a_1 r_1 + \cdots + a_n r_n \; ; \; \forall r_i \in R; i = 1, \cdots, n\}$$

is the ideal generated by $n-$element set $\{a_1, \cdots, a_n\}$. In forming those sums we are taking n arbitrary elements from the ring R in each summation.

Definition 4.6.5. *Cosets of Ideal* I *with respect to group* $(R, +)$ *: These cosets are defined in usual group notion as* $I + r$*, where* $r \in R$*.*

Note that $(R, +)$ is commutative. So there is no difference between right and left cosets.

Definition 4.6.6. *Quotient Group* R/I *: This quotient is defined in a natural way for construction of commutative group* $(R/I, \oplus)$ *by*

$$(I + r) \oplus (I + r') = I + (r + r') \tag{4.6.1}$$

Zero element of this quotient group is $I + 0 = I$

4.7. Quotient Rings

Definition 4.7.1. *Group Natural Map : Is defined as* $\pi : (R, +) \longrightarrow (R/I, \oplus)$ *such that* $r \mapsto I + r$.

In this sense π is a surjective group homomorphism. Now we are ready to grasp the concept of quotient ring by defining a multiplication in group $(R/I, \oplus)$ to construct a minimum structure of a semi-group

Definition 4.7.2. *Semi-group* $(R/I, \otimes)$*: Define multiplication* $\otimes$ *as*

$$(I + r) \otimes (I + r') = I + (rr') \tag{4.7.1}$$

It is straightforward to check that the resulting structure satisfies a semi-group structure. For the next definition to be correct, we have to assume that I is a **proper** ideal of R.

Definition 4.7.3. *Quotient Ring* $(R/I, [\oplus, \otimes])$ *: Quotient structure constructed by hinging commutative group* $(R/I, \oplus)$ *and semi-group* $(R/I, \otimes)$ *shows structure of a ring. It is called quotient ring or* ***residue*** *of* R ***modulo*** *ideal* I.

Members of the quotient rings constructed in this way are in the form of $I + r$. Note that the natural group quotient map now can be extended as the natural ring map from ring R to quotient ring R/I. and we have,

$$\pi(r)\,\pi(r') = \pi(rr') \tag{4.7.2}$$

where, $\pi : r \longmapsto I + r$. π is a surjective **ring** homomorphism. Also note that the right multiplicative semi-group cosets Ir has **not** anything to do with the quotient ring. If $(R, \cdot)$ is a monoid then one could show that $(R/I, \otimes)$ is a unitary ring, by checking that $I + 1$ is the unity of monoid. Remember, I is a proper ideal of R and hence $1 \notin I$.

Definition 4.7.4. *Maximal Ideal : Ideal* I *in a ring* R *is a maximal ideal if for any other ideal* J *in* R *we have* $J \subseteq I \subseteq R$.

Definition 4.7.5. *Prime Ideal : Ideal* I *in a ring* R *is said to be a prime ideal if it is a proper ideal of* R *and if* $ab \in I$ *implies* $a \in I$ *or* $b \in I$.

Definition 4.7.6. *Principal Ideal Domain (PID) : An integral domain in which every ideal is a principal ideal is called a principal ideal domain.*

- every subring of a field is a domain.
- for every domain there is a field containing domain as a subring. This containing field is called **fraction field** of the domain.
- a subfield of a ring R is a subring that is a field.

Definition 4.7.7. *Quotient Ideal: Suppose* I *and* J *are two ideals in a ring* R, *such that* $J \subseteq I$

4.8. Modules

Definition 4.8.1. *R-Module : An R-Module is a structure built of a duet of sets and a quartet of binary operations,* $\left([\mathbf{M}, R], \left[\vec{+}, \odot, +, \cdot\right]\right)$ *where, the substructure* $\left(\mathbf{M}, \vec{+}\right)$ *is an abelian (commutative) group, and the substructure* $(R, [+, \cdot])$ *is a ring. Additionally,*

(1) $\odot$ *is the action of Abelian group* $(R, \cdot)$ *on the set* $\mathbf{M}$.
(2) $\mathbf{M}$ *is an* R*-set with respect to* $\odot$ *action. That is, as you might remember from the definition (4.3.2) of a* G*-set,* $\forall x \in \mathbf{M}$ *we have* $(r_1 \cdot r_2) \odot x = r_1 \odot (r_2 \odot x)$*;* $\forall r_1$ *and* $\forall r_2 \in R$ *and* $e \odot x = x \; \forall x \in \mathbf{M}$*, where* e *is the unity of* $(R, \cdot)$.
(3) *we have* $(a + b) \odot x = a \odot x \vec{+} b \odot x$.

Note that we have differentiated between addition in group $\left(\mathbf{M}, \vec{+}\right)$ with addition in the ring $(R, [+, \cdot])$. Action $\odot$ maps addition $+$ in the ring to addition $\vec{+}$ in the group $\mathbf{M}$. You may notice that all properties of a G-set transfers to an R-Module by action $\odot$ of R on $\mathbf{M}$.

Definition 4.8.2. *Left R-Module : In the previous definition the action* $\odot$ *of* $(R, \cdot)$ *is defined at the* ***left*** *of the set* $\mathbf{M}$*. Hence, it is a* ***left*** *R-Module.*

Normally by an R-Module we mean a left R-Module.

Definition 4.8.3. *Right R-Module : If the action* $\odot$ *of group* $(R, \cdot)$ *is defined on the* ***right*** *of* $\mathbf{M}$ *then the R-Module is a* ***right*** *R-Module.*

Definition 4.8.4. *: page 36 AdvAbstAlgebra.pdf*

Definition 4.8.5. *Commutative R-Module : An R-Module is told to be a commutative R-Module when it is both left R-Module and right R-Module.*

Definition 4.8.6. *:*

4.9. R-Algebras

Definition 4.9.1. *R-Algebra : Assume* R *is a commutative ring then the duet of sets and a quintet array of binary operations,* $\left([\mathbf{M}, R], \left[\bullet, \vec{+}, \odot, +, \cdot\right]\right)$ *where*

(1) $\left([\mathbf{M}, R], \left[\vec{+}, \odot, +, \cdot\right]\right)$ *is an R-Module and*
(2) $(\mathbf{M}, \bullet)$ *is a commutative semigroup.*
(3) $\left(\mathbf{M}, \left[\vec{+}, \bullet\right]\right)$ *is a commutative ring and*
(4) $\forall r \in R$ *we have* $(x_1 \bullet x_2) \odot r = x_1 \bullet (x_2 \odot r)$*;* $\forall x_1$ *and* $\forall x_2 \in \mathbf{M}$.

Hence, in an R-Algebra two rings $(R, [+, \cdot])$ and $\left(\mathbf{M}, \left[\vec{+}, \bullet\right]\right)$ become hinged through the action $\odot$.

Definition 4.9.2. *Associative division R-Algebra : If* $(\mathbf{M}, \bullet)$ *assumed to be a group rather than a semigroup then R-Algebra is called an associative R-Algebra.*

Definition 4.9.3. *Inner Products in R-Module: It is possible to define a binary operation* $\circ : \mathbf{M} \times \mathbf{M} \longrightarrow R$ *with certain properties to create a duet of sets and a quintet array of binary operations,* $\left([\mathbf{M}, R], \left[\circ, \vec{+}, \odot, +, \cdot\right]\right)$.

(1) $\left([\mathbf{M}, R], \left[\vec{+}, \odot, +, \cdot\right]\right)$ *is an R-Module and*
(2) $\circ$ *is commutative.*
(3) $\circ$ *is distributive over* $\vec{+}$.
(4) $\forall r \in R$ *we have* $(x_1 \circ x_2) \odot r = x_1 \circ (x_2 \odot r)$*;* $\forall x_1$ *and* $\forall x_2 \in \mathbf{M}$.

We should distinguish carefully between R-Algebras and Inner Products in R-Modules, though they have an R-Module as a common part. The inner product in R-Module structure has not those nice algebraic properties of an R-Algebra, in being two hinged rings through the action of a group. For example, $(\mathbf{M}, \circ)$ is **not** a semigroup and $\left(\mathbf{M}, \left[\vec{+}, \circ\right]\right)$ is **not** a ring. Nevertheless it become more important when we create the similar structure of an inner product space on vector spaces later. We also should always use modifier R- to distinguish it with a similar structure, later we build on fields, where we use modifier K- to contrast it.

4.10. Fields

Definition 4.10.1. *:*

Definition 4.10.2. *Field : A field is a division ring* $(R, [+, \cdot])$ *where* $(R, \cdot)$ *is an abelian (commutative) group.*

In a field both addition and multiplication are commutative. It is customary to use the letter K as the set in the field in place of R and leave the letter R to be used only when a ring is involved. Hence, we show a field as $(K, [+, \cdot])$.

Definition 4.10.3. *:*

Definition 4.10.4. *:*

4.11. Vector Spaces

Definition 4.11.1. *K-Vector Space : A K-Vector Space is a structure built of a duet of sets and a quartet of binary operations,* $\left([\mathbf{V}, K], \left[\vec{+}, \odot, +, \cdot\right]\right)$ *where, the substructure* $\left(\mathbf{V}, \vec{+}\right)$ *is an abelian (commutative) group, and the substructure* $(K, [+, \cdot])$ *is a field. Additionally,*

(1) *$\odot$ is the action of Abelian group $(K, \cdot)$ on the set $\mathbf{V}$.*
(2) *$\mathbf{V}$ is a K-set with respect to $\odot$ action. That is, as you might remember from the definition (4.3.2) of a G-set, $\forall x \in \mathbf{V}$ we have $(r_1 \cdot r_2) \odot x = r_1 \odot (r_2 \odot x)$; $\forall r_1$ and $\forall r_2 \in K$ and $e \odot x = x$, $\forall x \in \mathbf{V}$, where e is the unity of $(K, \cdot)$.*
(3) *we have $(a + b) \odot x = a \odot x \vec{+} b \odot x$.*

Note that we have differentiated between addition in group $\left(\mathbf{V}, \vec{+}\right)$ with addition in the field $(K, [+, \cdot])$. Action $\odot$ maps addition $+$ in the field to addition $\vec{+}$ in the group $\mathbf{V}$. You may notice that all properties of a G-set transfers to a K-Vector Space by action $\odot$ of K on $\mathbf{V}$.

Definition 4.11.2. *Functional : A mapping f from $K-$vector space $\mathbf{V}$ to its underlaying field K, i.e., $f : \mathbf{V} \longrightarrow K$ is called a functional.*

Definition 4.11.3. *:*

4.12. K-Algebras

Definition 4.12.1. *K-Algebra : Assume K is a field then the duet of sets and a quintet array of binary operations,* $\left([\mathbf{V}, K], \left[\bullet, \vec{+}, \odot, +, \cdot\right]\right)$ *where*

(1) $\left([\mathbf{V}, K], \left[\vec{+}, \odot, +, \cdot\right]\right)$ *is a K-vector space and*

(2) $(\mathbf{V},\, \bullet\,)$ *is a commutative semigroup.*
(3) $\left(\mathbf{V},\, \left[\vec{+},\, \bullet\right]\right)$ *is a field and*
(4) $\forall r \in K$ *we have* $(x_1 \bullet x_2) \odot r = x_1 \bullet (x_2 \odot r)$*;* $\forall x_1$ *and* $\forall x_2 \in \mathbf{V}$.

Hence, in a K-Algebra two fields $(K,\, [+,\, \cdot\,])$ and $\left(\mathbf{V},\, \left[\vec{+},\, \bullet\right]\right)$ become hinged through the action $\odot$.

Definition 4.12.2. *Associative division K-Algebra : If* $(\mathbf{V},\, \bullet\,)$ *assumed to be a group rather than a semigroup then K-Algebra is called an associative K-Algebra.*

Definition 4.12.3. *Inner Products in K-Vector Space: It is possible to define a binary operation* $\circ : \mathbf{V} \times \mathbf{V} \longrightarrow K$ *with certain properties to create a duet of sets and a quintet array of binary operations,* $\left([\mathbf{V}, K]\,,\, \left[\circ,\, \vec{+},\, \odot,\, +,\, \cdot\,\right]\right)$.

(1) $\left([\mathbf{V}, K]\,,\, \left[\vec{+},\, \odot,\, +,\, \cdot\,\right]\right)$ *is a K-vector space and*
(2) $\circ$ *is commutative.*
(3) $\circ$ *is distributive over* $\vec{+}$.
(4) $\forall r \in K$ *we have* $(x_1 \circ x_2) \odot r = x_1 \circ (x_2 \odot r)$*;* $\forall x_1$ *and* $\forall x_2 \in \mathbf{V}$.

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